Axioms for higher category theory

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7th Workshop on Formal Topology
Venice
15 April 2026

The idea of ∞-categories

In homotopy-coherent mathematics, we systematically replace *sets* with (homotopy) *types*.

In set-level mathematics, a category C has

1. A set $\mathrm{Ob}(C)$ of objects
2. For $x, y : \mathrm{Ob}(C)$, a set $C(x, y)$ of morphisms
3. A unital, associative composition operation $C(y, z) \times C(x, y) \to C(x, z)$

Intuitively, in homotopy-coherent mathematics, an ∞-category has

1. A type $\mathrm{Ob}(C)$ of objects
2. For $x, y : \mathrm{Ob}(C)$, a type $C(x, y)$ of morphisms
3. A **coherently** unital, associative composition operation $C(y, z) \times C(x, y) \to C(x, z)$

The importance of ∞-category theory

- In set-level mathematics, category theory is very useful, but one can get quite far without it.
- In homotopy-coherent mathematics, ∞-category theory is *indispensible*.

The rub

The basic ideas of ∞-category theory are clear, but the subject is famously difficult to learn and work with.

Problem: how to explain the subject, in a way that is simultaneously rigorous and accessible?

More important than defining what ∞-categories *are*, we must explain how to work with them.

Previous work in this direction includes the theory of ∞-cosmoi (Riehl–Verity) and simplicial type theory (Riehl–Shulman, Buchholtz–Gratzer–Weinberger)

Good news

Practicioners today know how to work with ∞-categories 'synthetically', in a way that is mostly rigorous.

How to make formal sense of this practice?

Axioms for ∞-category theory

In a work-in-progress book project, Cisinski, Cnossen, Nguyen, and Walde develop ∞-category theory from basic principles.

Our contribution: explain what CCNW are doing in the context of HoTT (and maybe clarify some aspects).

NB: We expect many axioms of CCNW to be redundant, so omit them. But this has not been checked carefully.

The setting

We push back on the widespread idea that types must be interpreted as something more general than ∞-groupoids in order to do synthetic category theory.

We work in homotopy type theory with its *standard* interpretation: types are spaces / ∞-groupoids.

We have:

- Σ
- Π (with funext)
- $=$
- standard finite types
- $\mathbb{N}$
- propositional truncation
- univalent universe U closed under the above.

The wild category of synthetic (∞-)categories

We postulate the following:

- We have a type Cat. Elements of Cat are called *categories*.
- For C, D : Cat, we have a small type $\mathrm{Map}(C, D) : \mathrm{U}$. Elements of $\mathrm{Map}(C, D)$ are called *functors*.
- Given C : Cat, we have $\mathrm{id}_C : \mathrm{Map}(C, C)$. This is called the identity functor.
- Given $F : \mathrm{Map}(C, D)$, $G : \mathrm{Map}(D, E)$, we have $G \circ F : \mathrm{Map}(C, E)$. This is called functor composition.

We also extend HoTT with the following judgments:

- For $F : \mathrm{Map}(A, B)$, we have *judgmental* equality

$$\mathrm{id}_B \circ F \equiv F \equiv F \circ \mathrm{id}_A$$

- For $F : \mathrm{Map}(A, B)$, $G : \mathrm{Map}(B, C)$, $H : \mathrm{Map}(C, D)$, we have

$$H \circ (G \circ F) \equiv (H \circ G) \circ F$$

Coherence for functor composition

Judgmental associativity and unit laws means that an infinite tower of higher coherences (pentagonator etc) are automatic.

We conjecture that it would be sufficient to postulate an axiom schema of weak coherences for functor composition.

But working explicitly with those seems difficult, and intended semantics justifies strict associativity and unit laws.

The form of our postulates

Heuristically, *just adding postulates* is problematic:
what if we are missing higher coherences?

Postulating judgment rules is even worse.

From now on, all[1] postulates are of the form "we have an element of P" where P is a proposition.

(The reason this *ought* to work is that Cat has very few (two) automorphisms.)

[1] With one exception.

Invertible functors

For f : $\mathrm{Map}(C, D)$, the following type is a proposition:

$$\Sigma(g\,h : \mathrm{Map}(D, C)) \times (f \circ g = \mathrm{id}_D) \times (h \circ f = \mathrm{id}_C)$$

and it is equivalent to either one of two conditions

1. For all X : Cat, the map of types
   $f \circ - : \mathrm{Map}(X, C) \to \mathrm{Map}(X, D)$ is an equivalence
2. For all X : Cat, the map of types
   $- \circ f : \mathrm{Map}(D, X) \to \mathrm{Map}(C, X)$ is an equivalence

We denote this proposition by $\mathrm{isEquiv}(f)$: U.

Denote $\Sigma(f : \mathrm{Map}(C, D)) \times \mathrm{isEquiv}(f)$ by $C \simeq D$.

Univalence of Cat

Postulate: for any C : Cat, the following type is contractible.

$$\Sigma(D : \mathrm{Cat}) \times (C \simeq D)$$

Equivalently, we have

$$(C \simeq D) \simeq (C = D)$$

The terminal category

We postulate:

- we have a category $1 : \mathrm{Cat}$,
- for all $C : \mathrm{Cat}$, the type $\mathrm{Map}(C, 1)$ is contractible.

For short, we say 'Cat has a terminal object'.

Because Cat is univalent, this is naturally a proposition.

Objects of a category

The type $\mathrm{Map}(1, C)$ is denoted by $\mathrm{Ob}(C)$.

Elements of $\mathrm{Ob}(C)$ are called *objects* (of C).

$F : \mathrm{Map}(C, D)$ induces

$$F_{\mathrm{Ob}} : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$$

given by $F_{\mathrm{Ob}}(c) := F \circ c$.

Pullbacks of categories

Given A, B, C : Cat, f : Map(B, A), g : Map(C, A), we postulate:

- a category $B \times_A^{f,g} C$: Cat, or $B \times_A C$ for short
- a functor fst : Map$(B \times_A C, B)$
- a functor snd : Map$(B \times_A C, C)$
- an identification $\alpha : f \circ \mathrm{fst} = g \circ \mathrm{snd}$
- such that for all X : Cat, the map

$$\mathrm{Map}(X, B \times_A C) \to \Sigma(b : \mathrm{Map}(X, B)(c : \mathrm{Map}(X, C)(f \circ b = g \circ c)$$
$$p \mapsto (\mathrm{fst} \circ p, \mathrm{snd} \circ p, \mathrm{ap}_{-\circ p}(\alpha))$$

  is an equivalence of types.

For short, we say that 'Cat has pullbacks'.

Product categories

In particular Cat has binary products $A \times B$, given by $A \times_1 B$.

Note $\mathrm{Ob}(A \times B) \simeq \mathrm{Ob}(A) \times \mathrm{Ob}(B)$.

Functor categories

Given A, B : Cat, we postulate:

- a category $\mathrm{Fun}(A, B)$: Cat
- a functor $\mathrm{ev} : \mathrm{Map}(\mathrm{Fun}(A, B) \times A, B)$
- such that for all X : Cat, the map

$$\begin{aligned} \mathrm{Map}(X, \mathrm{Fun}(A, B)) &\to \mathrm{Map}(X \times A, B) \\ p &\mapsto \mathrm{ev} \circ (p \times A) \end{aligned}$$

  is an equivalence of types.

For short, we say that 'Cat has exponential objects'.

Exercise: construct equivalence $\mathrm{Ob}(\mathrm{Fun}(A, B)) \simeq \mathrm{Map}(A, B)$.

Morphisms and the inverval

We postulate:

1. A category $\mathbb{I}$: Cat
2. An equivalence $e : \mathrm{Ob}(\mathbb{I}) \simeq \{0, 1\}$

We denote $e^{-1}(0)$, $e^{-1}(1)$ simply by 0, 1.

A functor $f : \mathrm{Map}(\mathbb{I}, C)$ is called a *morphism*.

The domain of f is $\mathrm{dom}(f) := f_{\mathrm{Ob}}(0)$;
the codomain is $\mathrm{cod}(f) := f_{\mathrm{Ob}}(1)$.

The fibre of $(\mathrm{dom}, \mathrm{cod}) : \mathrm{Map}(\mathbb{I}, C) \to \mathrm{Ob}(C) \times \mathrm{Ob}(C)$ over (x, y) is denoted $C(x, y)$.

Exercise: $F : \mathrm{Map}(C, D)$ induces $F_{\mathrm{mor}} : C(x, y) \to D(F_{\mathrm{Ob}}x, F_{\mathrm{Ob}}y)$.
For $x : \mathrm{Ob}(C)$ we have $\mathrm{id}_x : C(x, x)$.

Functors into $\mathbb{I}$

$\mathrm{Ob}(\mathbb{I})$ has a linear order where $0 \leq 1$.

Say a function $p : \mathrm{Ob}(C) \to \mathrm{Ob}(\mathbb{I})$ is *monotone* if for all $f : \mathrm{Map}(\mathbb{I}, C)$, we have $p(\mathrm{dom}(f)) \leq p(\mathrm{cod}(f))$.

We postulate:

- For all C : Cat, the function

$$\mathrm{Map}(C, \mathbb{I}) \to \mathrm{Ob}(\mathbb{I})^{\mathrm{Ob}(C)}$$
$$F \mapsto F_{\mathrm{Ob}}$$

  is (-1)-truncated, and its image consists precisely of monotone maps.

The category $[n]$

We define $[n] : \mathsf{Cat}$ for $n : \mathbb{N}$ by recursion:

- $[0] := 1$
- $[n+1] := \mathsf{Fun}([n], \mathbb{I})$

Functors into $[n]$

One can show:

- Every hom-type of $[n]$ is a proposition.
- $[n]$ 'is' a finite linear order, isomorphic to $\{0 \leq 1 \cdots \leq n\}$.
- For every C : Cat, the type $\mathrm{Map}(C, [n])$ is equivalent to the set of monotone functions

$$\mathrm{Ob}(C) \to \{0 \leq 1 \cdots \leq n\}.$$

The Segal axiom

We postulate: the following square of categories is a pushout.

$$\begin{array}{ccc} 1 & \xrightarrow{0} & \mathbb{I} \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 12} \\ \mathbb{I} & \xrightarrow[01]{} & [2] \end{array}$$

Explicitly, this means that for all C : Cat, the canonical map

$$\mathsf{Map}([2], C) \to \mathsf{Map}(\mathbb{I}, C) \times^{\mathsf{cod},\mathsf{dom}}_{\mathsf{Map}(1,C)} \mathsf{Map}(\mathbb{I}, C)$$

is an equivalence of types.

Composition of morphisms

Let $\mathrm{Tri}(C)$ denote the type of 'triangles' in C:

$$\mathrm{Tri}(C) := \Sigma(x\,y\,z : \mathsf{Ob}(C)) \times C(x, y) \times C(y, z) \times C(x, z)$$

We have a triangle in [2] given by $(0, 1, 2, -, -, -)$.

Functors act on triangles, so we have a function

$$\mathsf{Map}([2], C) \to \mathrm{Tri}(C).$$

Denote fibre over t by $\mathsf{Comm}(t)$.

The Segal axiom means: given $x, y, z : \mathsf{Ob}(C)$, $f : C(x, y)$, $g : C(y, z)$, the type $\Sigma(h : C(x, z)) \times \mathsf{Comm}(f, g, h)$ is contractible.

This defines $g \circ f : C(x, z)$ and an equivalence

$$\mathsf{Comm}(f, g, h) \simeq (h = g \circ f).$$

$[n]$ is the colimit of its spine

We can now prove the following: for every $C : \mathsf{Cat}$, $n : \mathbb{N}$, 'the map'

$$\mathsf{Map}([n], C) \to \Sigma(a : \mathsf{Ob}(C)^{\mathsf{Fin}(n+1)}) \times \Pi_{i:\mathsf{Fin}(n)} C(a(i), a(i+1))$$

is an equivalence of types.

In other words:

- $[n]$ is freely generated by a sequence of n composable maps
- $[n]$ is the colimit of $\mathbb{I} \xleftarrow{1} 1 \xrightarrow{0} \mathbb{I} \xleftarrow{1} \cdots \xrightarrow{0} \mathbb{I}$

Proof sketch that $[n]$ is colimit of its spine

We induct on n. For $n = 0$ there is nothing to prove.

Assuming $n : \mathbb{N}$ such that $[n]$ is the colimit of its spine, since Cat has exponential objects we get that $[n] \times \mathbb{I}$ is the colimit of

$$\mathbb{I} \times \mathbb{I} \leftarrow 1 \times \mathbb{I} \rightarrow \mathbb{I} \times \mathbb{I} \leftarrow 1 \times \mathbb{I} \cdots \rightarrow \mathbb{I} \times \mathbb{I}.$$

This diagram retracts to

$$\Delta^2 \leftarrow 1 \rightarrow \mathbb{I} \leftarrow 1 \cdots \rightarrow \mathbb{I}$$

and $[n] \times \mathbb{I}$ retracts to $[n+1]$.

Moreover, the retraction / section maps are compatible in an appropriate sense.

Conclude by appealing to the fact that equivalences of types are closed under retracts.

What goes into the proof

In the proof we used that

- The ‘arrow’ category of Cat has unital and associative composition.
- Given a ‘graph’ (A, R) with $A : \mathrm{U}$, $R : A \to A \to \mathrm{U}$, we may talk of (A, R)-shaped diagrams in Cat, and of morphisms between such diagrams.
- Morphisms of (A, R)-shaped diagrams compose, composition is associative and unital.
- Every category C : Cat determines a constant diagram of shape (A, R). The constant diagram is functorial in C. A cocone is a diagram morphism into a constant diagram.
- We have a good understanding of cocones to $[n + 1]$, $[n] \times \mathbb{I}$.

Associativity

Given $x, y, z, w : \mathrm{Ob}(C)$, $f : C(x, y)$, $g : C(y, z)$, $h : C(z, w)$, we want to build an identification

$$h \circ (g \circ f) = (h \circ g) \circ f$$

in $C(x, w)$.

If C is [3] then this is trivial: the hom-set $C(x, w)$ is contractible.

We obtain the general associator by observing that f, g, h are induced by a functor $p : \mathrm{Map}([3], C)$ and that this p respects morphism composition.

Invertible morphisms

Given $f : C(x, y)$, we say that f is invertible if any of the following equivalent conditions hold.

1. f has a left inverse and a right inverse
2. for all $z : \mathrm{Ob}(C)$, $f \circ - : C(z, x) \to C(z, y)$ is an equivalence of types
3. for all $z : \mathrm{Ob}(C)$, $- \circ f : C(y, z) \to C(x, z)$ is an equivalence of types.

We denote the type of invertible morphisms by $x \simeq y$.

We say that C is *groupoid* if all its morphisms are invertible.

The Rezk axiom: every category is univalent

We postulate: for every $C : \mathrm{Cat}$, $x : \mathrm{Ob}(C)$, the type

$$(y : \mathrm{Ob}(C)) \times (x \simeq y)$$

is contractible. Equivalently,

$$(x \simeq y) \simeq (x = y)$$

The triangulation of the square

We postulate: the following square is a pushout.

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{02} & [2] \\ {\scriptstyle 02}\big\downarrow & & \big\downarrow{\scriptstyle (011,001)} \\ [2] & \xrightarrow[(001,011)]{} & \mathbb{I} \times \mathbb{I} \end{array}$$

$\mathbb{I}$ detects equivalences

We postulate: given $F : \mathsf{Map}(C, D)$ such that

$$F \circ - : \mathsf{Map}(\mathbb{I}, C) \to \mathsf{Map}(\mathbb{I}, D)$$

is an equivalence of types, F is invertible.

Equivalently: a functor $F : \mathsf{Map}(C, D)$ is invertible if and only if it is

- fully faithful, i.e. for all $x\, y : \mathsf{Ob}(C)$,

  $$F_{\mathrm{mor}} : C(x, y) \to C(F_{\mathsf{Ob}} x, F_{\mathsf{Ob}} y)$$

  is an equivalence of types
- surjective, i.e. $F_{\mathsf{Ob}} : \mathsf{Ob}(C) \to \mathsf{Ob}(D)$ is a surjection of types.

Full subcategories exist

Given C : Cat and a family $P : \mathrm{Ob}(C) \to \mathrm{U}$ of small propositions, we postulate

- a category C_P : Cat
- a functor $i : \mathrm{Map}(C_P, C)$
- such that for all D : Cat, the map

$$i \circ - : \mathrm{Map}(D, C_P) \to \mathrm{Map}(D, C)$$

  is (-1)-truncated, and its image consists precisely of those $f : \mathrm{Map}(D, C)$ such that for all $d : D$, $P(f_{\mathrm{Ob}} d)$ holds.

Exercise: a fully faithful functor into C is determined by its image.

Coproducts of categories

Given a small type X : U, we postulate:

- Cat has X-indexed *coproducts*
- X-indexed coproducts in Cat are *extensive*:

$$\mathrm{Cat}/(\sqcup_{x:X} C_x) \simeq \Pi_{x:X} \mathrm{Cat}/C_x.$$

- the coproduct $\sqcup_X 1$ is a groupoid
- the map $X \to \mathrm{Ob}(\sqcup_X 1)$ is an equivalence of types.

Exercise: if C : Cat is a groupoid, then $\sqcup_{\mathrm{Ob}(C)} 1 \to C$ is invertible.
So groupoids correspond to (small) types.

Cat has pushouts

We postulate that Cat has pushouts.

Left fibrations

We say that a functor $p : \mathrm{Map}(E, B)$ is a *left fibration* if it is right orthogonal against $0 : 1 \to \mathbb{I}$,
i.e. if the following square of types is a pullback.

$$\begin{array}{ccc}
\mathrm{Map}(\mathbb{I}, E) & \xrightarrow{p\circ -} & \mathrm{Map}(\mathbb{I}, B) \\
{\scriptstyle \mathrm{dom}}\downarrow & & \downarrow{\scriptstyle \mathrm{dom}} \\
\mathrm{Map}(1, E) & \xrightarrow[p\circ -]{} & \mathrm{Map}(1, B)
\end{array}$$

Subuniversal left fibrations

We say that a left fibration $p : \mathrm{Map}(E, B)$ is *subuniversal* if for every C : Cat, the map

$$\begin{aligned} \mathrm{Map}(C, B) &\to \Sigma(X : \mathrm{Cat}) \times \mathrm{Map}(C, X) \\ f &\mapsto (C \times_B^{f,p} E, \mathrm{fst}) \end{aligned}$$

is (-1)-truncated, and its image consists of left fibrations $q : X \to C$ such that for all $c : C$ there is $b : B$ with $\mathrm{fib}_q(c) \simeq \mathrm{fib}_p(b)$.

Think of a subuniversal left fibration as a 'category of (some) types'

Left fibration classifiers

We postulate: every left fibration is a pullback of a subuniversal left fibration.

We should have:

- the type of subuniversal left fibrations is equivalent to the set of small subtypes of U.
- A left fibration $p : E \to B$ is subuniversal if and only if it is *directed univalent*, i.e. for all $x, y : B$, the transport map

$$B(x, y) \to (\mathrm{fib}_p(x) \to \mathrm{fib}_p(y))$$

  is invertible.

Example: Limits

We say that $c : \mathrm{Ob}(C)$ is *terminal* if for all $d : \mathrm{Ob}(C)$, the type $C(d, c)$ is contractible.

Given $A : \mathrm{Map}(J, C)$, the category $C \downarrow A : \mathrm{Cat}$ of *cones* over A is given by the following pullback.

$$
\begin{array}{ccc}
C \downarrow A & \longrightarrow & C \\
\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle (\Delta, D \circ !)} \\
\mathrm{Fun}(J \times \mathbb{I}, C) & \xrightarrow[(\mathrm{dom},\mathrm{cod})]{} & \mathrm{Fun}(J, C) \times \mathrm{Fun}(J, C)
\end{array}
$$

A *limit cone* for A is a terminal object of $C \downarrow A$.

Example: Natural transformations

Given functors $F, G : \mathsf{Map}(C, D)$, a *natural transformation* is a morphism between the corresponding objects of $\mathsf{Fun}(C, D)$.

Natural transformations induce *naturality square*, since

$$\mathsf{ev} : \mathsf{Fun}(C, D) \times C \to D$$

sends commutatative squares to commutative squares.

Expected results

We expect to be able to prove:

- The Yoneda lemma, in the form 'the inclusion of an initial object is left cofinal' (following Haugseng)
- Every category C : Cat has an opposite category C^{op} : Cat, characterised by having a 'balanced coupling' $\mathrm{Tw}(C) \to C^{\mathsf{op}} \times C$ in the sense of Lurie (following CCNW)
- Left Kan extension of left fibration exists and is computed pointwise
- Filtered colimits of types commute with finite limits
- Hom-types of nice enough colimits of categories admit explicit descriptions
- The full image factorisation of any functor f : Map(C, D) admits a description similar to Rijke's join construction
- Cat has enough subuniversal *cocartesian* fibrations ('category of (some) categories')

Axioms

The exact collection of axioms listed here is not so important.
What is important is:

- All axioms are propositions, except for enumeration of $\mathrm{Ob}(\mathbb{I})$.
- The axioms are fairly easy to state and have clear meaning.
- The list of axioms is fairly short.
- We fruitfully leverage HoTT in its standard interpretation.
- Developing higher category theory directly from these axioms seems feasible, *even in existing proof assistant (Agda, Rocq)*!

References

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