PATH SPACES OF PUSHOUTS

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ABSTRACT. Working in homotopy type theory, we describe path spaces of pushouts as sequential colimits of pushouts. This gives a higher Seifert–van Kampen theorem. We demonstrate the utility of this description by showing that any pushout of 0-types is a 1-type, settling a long-standing open question in homotopy type theory. Our construction can be interpreted in any higher category with pullbacks, finite colimits satisfying descent, and sequential colimits.

1. INTRODUCTION

Many important questions in homotopy theory concern iterated path spaces of higher inductive types, so it is clearly desirable to have a useful description of such path spaces. In [3], Brunerie gave such a description for the loop space of the suspension of a pointed connected type. In [6], Kraus and von Raumer gave a universal property for path spaces of general coequalisers and pushouts. This gives a description of the path space for general non-recursive higher inductive types, but because the description is recursive, it does not directly give a useful description of iterated path spaces.

In this work, we build on the results of Kraus and von Raumer by presenting path spaces of pushouts as sequential colimits of pushouts. Since sequential colimits commute with path spaces [8], this means we can recursively understand higher path spaces of pushouts.

Our construction can be understood as building a sequence of approximations to the path spaces of pushouts, converging to an exact description. What makes this useful is a combination of two features: the definition of the sequence of approximations is relatively simple, and each term forms a relatively good approximation of the colimit. We demonstrate this by giving relatively simple proofs of a host of results about path spaces of pushouts, including the fact that pushouts of 0-types are 1-types. This fact can be understood as a coherence theorem for higher groupoids and has eluded homotopy type theorists for over a decade. Our result is a significant improvement of the Seifert–van Kampen theorem [4], which describes the set-truncated path spaces of pushouts.

2. PATH SPACES OF PUSHOUTS ARE SEQUENTIAL COLIMITS

We now present the main construction. Consider a general span, represented as a pair of types $A : \mathcal{U}$, $B : \mathcal{U}$ together with a relation[1] $R : A \to B \to \mathcal{U}$. Write $A +_R B$ for the pushout of this span, and $\mathsf{inl} : A \to A +_R B$, $\mathsf{inr} : B \to A +_R B$ for the inclusions of A and B. Our aim is to describe path types of the form $\mathsf{inl}(a_0) = \mathsf{inl}(a)$, $\mathsf{inl}(a_0) = \mathsf{inr}(b)$, and $\mathsf{inr}(b) = \mathsf{inr}(b')$. Without loss of generality, we will concentrate on the first two; the third is understood symmetrically. By [6], these types are in a precise sense freely generated by points $\mathsf{refl} : (a_0 : A) \to \mathsf{inl}(a_0) = \mathsf{inl}(a_0)$ and equivalences $(a_0\, a : A)\,(b : B) \to R(a, b) \to (\mathsf{inl}(a_0) = \mathsf{inl}(a)) \simeq (\mathsf{inl}(a_0) = \mathsf{inr}(b))$.

Intuitively, a proof of $\mathsf{inl}(a_0) = \mathsf{inl}(a)$ is given by a zigzag of edges $r : R(a', b')$, starting at a_0 and ending at a, and an element of $\mathsf{inl}(a_0) = \mathsf{inr}(b)$ is given by a zigzag starting at a_0 and ending at b. We will show that $\mathsf{inl}(a_0) = \mathsf{inl}(a)$ is equivalent to a sequential colimit, $a_0 \rightsquigarrow_\infty a :\equiv \mathrm{colim}_{t\to\infty}\, a_0 \rightsquigarrow_t a$, where intuitively $a_0 \rightsquigarrow_t a$ corresponds to zigzags of length at most t, and similarly $a_0 \rightsquigarrow_\infty b :\equiv$

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[1]Note that this 'relation' is proof-relevant, i.e. valued in arbitrary types.

$\operatorname{colim}_{t\to\infty} a_0 \rightsquigarrow_t b$. We use the notational convention that $a_0 \rightsquigarrow_t a$ is indexed by even $t \geq 0$ and $a_0 \rightsquigarrow_t b$ is indexed by odd $t \geq -1$. We take $a_0 \rightsquigarrow_0 a :\equiv (a_0 = a)$ as there is only a length zero zigzag from a_0 to itself, and $a_0 \rightsquigarrow_{-1} b :\equiv \bot$ the empty type.

We will define maps $\iota_{t,a_0,a} : (a_0 \rightsquigarrow_t a) \to (a_0 \rightsquigarrow_{t+2} a)$, $\iota_{t,a_0,b} : (a_0 \rightsquigarrow_t b) \to (a_0 \rightsquigarrow_{t+2} b)$, and $\cdot_t r : (a_0 \rightsquigarrow_t a) \to (a_0 \rightsquigarrow_{t+1} b)$, $\cdot_t \overline{r} : (a_0 \rightsquigarrow_t b) \to (a_0 \rightsquigarrow_{t+1} a)$ for $r : R(a,b)$. The latter two maps we write in postfix form, so that $p \cdot_t r : a_0 \rightsquigarrow_{t+1} b$ for $p : a_0 \rightsquigarrow_t a$. These maps will fit into the following commutative diagram for $r : R(a,b)$, expressing that $p \cdot r \cdot \overline{r} = p$ and $p \cdot \overline{r} \cdot r = p$.

$$\begin{tikzcd}
\cdots & a_0 \rightsquigarrow_t a \arrow[rr, "\iota_{t,a_0,a}"] \arrow[dr, "\cdot_t r"'] & & a_0 \rightsquigarrow_{t+2} a \arrow[dr, "\cdot_{t+2} r"] & \cdots \\
\cdots & & a_0 \rightsquigarrow_{t+1} b \arrow[ur, "\cdot_{t+1} \overline{r}"] \arrow[rr, "\iota_{t+1,a_0,b}"] & & a_0 \rightsquigarrow_{t+3} b & \cdots
\end{tikzcd} \tag{1}$$

The commutativity of the above diagram means something very simple: that we have a witness of commutativity for each triangle, with no further coherences. We now define the types $a_0 \rightsquigarrow_t a$ and $a_0 \rightsquigarrow_t b$ freely to ensure we have the data above. More precisely, suppose we have defined $a_0 \rightsquigarrow_t a$, $a_0 \rightsquigarrow_{t+1} b$, and $\cdot_t r$ for all $a : A$, $b : B$, $r : R(a,b)$. We then define $a_0 \rightsquigarrow_{t+2} a$ by the following pushout diagram.[2]

$$\begin{tikzcd}
(b : B) \times R(a,b) \times a_0 \rightsquigarrow_t a \arrow[r] \arrow[d] & (b : B) \times R(a,b) \times a_0 \rightsquigarrow_{t+1} b \arrow[d] \\
a_0 \rightsquigarrow_t a \arrow[r] & a_0 \rightsquigarrow_{t+2} a \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{2}$$

The left vertical map is projection to the third factor, and the top horizontal map is given by $\cdot_t r$ which we assume has already been defined. If we view this span as a relation, it relates $p : a_0 \rightsquigarrow_t a$ with a triple (b, r, q) if $q = p \cdot_t r$. We define $\iota_{t,a_0,a}$ to be the left inclusion in the diagram above. We define $\cdot_{t+1}\overline{r}$ using the right inclusion. We have $p \cdot_t r \cdot_{t+1} \overline{r} = \iota_{t,a_0,a} p$ for $p : a_0 \rightsquigarrow_t a$ by the commutativity of the pushout square above. We define $a_0 \rightsquigarrow_{t+2} b$ symmetrically, using the pushout square below.

$$\begin{tikzcd}
(a : A) \times R(a,b) \times a_0 \rightsquigarrow_t b \arrow[r] \arrow[d] & (a : A) \times R(a,b) \times a_0 \rightsquigarrow_{t+1} a \arrow[d] \\
a_0 \rightsquigarrow_t b \arrow[r] & a_0 \rightsquigarrow_{t+2} b \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

This finishes the definitions of $\rightsquigarrow_t$ as well as the maps ι, $\cdot_t r$. We can now also define $\rightsquigarrow_\infty$ simply as a sequential colimit. Note that we have an element $\mathsf{refl}_{a_0} : a_0 \rightsquigarrow_0 a_0$, which gives an element also of $a_0 \rightsquigarrow_\infty a_0$, which we write $\mathsf{refl}^\infty_{a_0}$.

Lemma 1. For any two sequences interleaved by zigzags as in Diagram 1, the diagram induces an equivalence between the sequential colimits, in this case between $a_0 \rightsquigarrow_\infty a$ and $a_0 \rightsquigarrow_\infty b$.

We write this equivalence as $\cdot r : (a_0 \rightsquigarrow_\infty a) \to (a_0 \rightsquigarrow_\infty b)$.

Proof. The map from $a_0 \rightsquigarrow_\infty a$ to $a_0 \rightsquigarrow_\infty b$ is given by functoriality of sequential colimits, from $\cdot_t r$ together with the proof that this is natural in t, obtained by pasting the two commuting triangles. The inverse is defined in the same way using $\cdot_t \overline{r}$. The proof that maps are inverse to each other is straightforward but requires a little path algebra for which we do not go into details. $\square$

[2]We use the notation $(x : X) \times Y(x)$ for the sigma-type $\Sigma_{x:X} Y(x)$.

Theorem 2. We have the following equivalences of types for $a_0, a : A$ and $b : B$.

$$(a_0 \rightsquigarrow_\infty a) \simeq (\mathsf{inl}(a_0) = \mathsf{inl}(a))$$
$$(a_0 \rightsquigarrow_\infty b) \simeq (\mathsf{inl}(a_0) = \mathsf{inr}(b))$$

Proof. Fix $a_0 : A$. Following [6], it is enough to show the following dependent elimination principle for $\rightsquigarrow_\infty$. Let $P : (a : A) \to (a_0 \rightsquigarrow_\infty a) \to \mathcal{U}$ and $Q : (b : B) \to (a_0 \rightsquigarrow_\infty b) \to \mathcal{U}$ with $d : P(a_0, \mathsf{refl}^\infty_{a_0})$ and $e_r : P(a, p) \simeq Q(b, p \mathbin{\text{▪}} r)$ for $r : R(a, b)$. We have to define $s_P(a, p) : P(a, p)$ and $s_Q(b, q) : Q(b, q)$ for each a, p, b, q, with proofs $s_P(a_0, \mathsf{refl}^\infty_{a_0}) = d$ and $e_r(s_P(a, p)) = s_Q(b, p \mathbin{\text{▪}} r)$. Given this, it follows that we have an initial object in the appropriate wild category, and by uniqueness of initial objects we get the desired equivalence.

The definition of s_P, s_Q and the desired paths is all straightforward using the universal properties of sequential colimits and of the pushout defining $a_0 \rightsquigarrow_{t+2} a$ and $a_0 \rightsquigarrow_{t+2} b$. □

By the definition of the equivalence in the theorem, $\mathsf{refl}^\infty_{a_0}$ corresponds to $\mathsf{refl}_{\mathsf{inl}(a_0)}$, and if $p : a_0 \rightsquigarrow_\infty b$ corresponds to $q : \mathsf{inl}(a_0) = \mathsf{inl}(a)$, then $p \mathbin{\text{▪}} r$ corresponds to $q \cdot \mathsf{glue}(r)$, where glue is the path constructor $A +_R B$.

3. Some consequences

We now discuss some consequences of the above description.

Lemma 3. Given A, B, R, and $a_0 : A$ as before, suppose that for some even t the map $\mathbin{\text{▪}}_t r$ is an equivalence for each $a : A$, $b : B$, $r : R(a, b)$. Then every map in Diagram 1 after this is also an equivalence. Thus $(a_0 \rightsquigarrow_t a) = (a_0 \rightsquigarrow_\infty a)$ and $(a_0 \rightsquigarrow_{t+1} b) = (a_0 \rightsquigarrow_\infty b)$. The analogous statement holds also if we assume instead that t is odd and each $\mathbin{\text{▪}}_t \overline{r}$ is an equivalence.

Proof. The map $\iota_{t,a_0,a}$ is an equivalence since it is the pushout of an equivalence. Hence so is $\mathbin{\text{▪}}_{t+1} \overline{r}$, by 2-out-of-3. By induction, all the maps are equivalences as claimed. We have $(a_0 \rightsquigarrow_t a) = (a_0 \rightsquigarrow_\infty a)$ since equivalences are closed under transfinite composition. □

Lemma 4. We have $(a_0 \rightsquigarrow_1 b) \simeq R(a_0, b)$ and $\mathbin{\text{▪}}_0 r$ corresponds to the transport map $(a_0 = a) \to R(a_0, b)$ for $r : R(a, b)$.

Proof. The pushout square describing $a_0 \rightsquigarrow_1 b$ has the empty type in one corner, so we just have $(a_0 \rightsquigarrow_1 b) \simeq (a : A) \times R(a, b) \times (a_0 = a) \simeq R(a_0, b)$. □

Lemma 5. The following pushout square describes $a_0 \rightsquigarrow_2 a$.

$$\begin{tikzcd}
(b : B) \times R(a, b) \times (a_0 = a) \arrow[r] \arrow[d] & (b : B) \times R(a, b) \times R(a_0, b) \arrow[d] \\
(a_0 = a) \arrow[r] & a_0 \rightsquigarrow_2 a \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

In this square, the top map duplicates $r : R(a, b)$, and the left map is the third projection.

Proof. Direct by Lemma 4 and the definition of $\rightsquigarrow_2$. □

Lemma 6. Suppose that for each $b : B$ the type $(a : A) \times R(a, b)$ is a proposition. Then $\mathbin{\text{▪}}_0 r$ is already an equivalence for each r. If instead we assume that for each $a : A$ the type $(b : B) \times R(a, b)$ is a proposition, then $\mathbin{\text{▪}}_1 \overline{r}$ is an equivalence for each r.

Proof. We prove the first part first. For $r : R(a, b)$, the map $\mathbin{\text{▪}}_0 r$ is given by transport $(a_0 = a) \to R(a_0, b)$. This is an equivalence by the fundamental theorem of identity types, using the assumption that $(a' : A) \times R(a', b)$ is a proposition.

Now suppose $(b : B) \times R(a,b)$ is a proposition for each $a : A$, and that we have $a : A$, $b : B$, $r : R(a,b)$. Consider the pushout square in Lemma 5. The left map is an equivalence, since $(b' : B) \times R(a,b')$ is an inhabited proposition. Since equivalences are closed under pushout, the right map is also an equivalence. The map $\cdot_1\overline{r}$ corresponds to the composite $R(a_0,b) \to (b' : B) \times R(a,b') \times R(a_0,b') \to (a_0 \rightsquigarrow_2 a)$. We just argued that the second map is an equivalence, and the first map is an equivalence since $(b' : B) \times R(a,b')$ is again an inhabited proposition. So the composite map is an equivalence, as desired. ☐

More concretely, the above gives the following well-known result.

Theorem 7. Consider the following pushout square.[3]

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d] & B \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A +_R B \end{tikzcd}$$

where the top map is an embedding. Then

(1) The map $\mathsf{inl} : A \to A +_R B$ is an embedding.
(2) The square is a pullback square.
(3) For $b, b' : B$, the path type $\mathsf{inr}(b) = \mathsf{inr}(b')$ is given by the following pushout square.

$$\begin{tikzcd} (a : A) \times R(a,b) \times (b = b') \arrow[r] \arrow[d] & (a : A) \times R(a,b) \times R(a,b') \arrow[d] \\ b = b' \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathsf{inr}(b) = \mathsf{inr}(b') \end{tikzcd}$$

Proof. The assumption that the map $R \to B$ is an embedding means that for each $b : B$, the type $(a : A) \times R(a,b)$ is a proposition. So in this case, the map $(a_0 \rightsquigarrow_0 a) \to (a_0 \rightsquigarrow_\infty a)$ is an equivalence. This means that inl is an equivalence on paths, i.e an embedding.

In the same way, since $(a_0 \rightsquigarrow_1 b) \to (a_0 \rightsquigarrow_\infty b)$ is an equivalence, we have that $\mathsf{glue} : R(a_0,b) \to \mathsf{inl}(a_0) = \mathsf{inr}(b)$ is an equivalence. This means precisely that the square is a pullback square.

Changing the roles of A and B, we have that $(b \rightsquigarrow_2 b') \to (b \rightsquigarrow_\infty b')$ is an equivalence, which gives the third part of the theorem. ☐

Lemma 8. Suppose given a span A, B, R such that both maps $R \to A$, $R \to B$ are 0-truncated.[4] Then all the maps in Diagram 1 are embeddings. Since embeddings are closed under transfinite composition, the maps from $\rightsquigarrow_t$ to $\rightsquigarrow_\infty$ are also embeddings, for each t.

A direct consequence is that the maps $\mathsf{inl} : A \to A +_R B$ and $\mathsf{inr} : B \to A +_R B$ are 0-truncated (since their actions on paths correspond to the embedding of $\rightsquigarrow_0$ in $\rightsquigarrow_\infty$) and that the cartesian gap map $R(a,b) \to (\mathsf{inl}(a) = \mathsf{inr}(b))$ is an embedding (corresponding to the embedding of $\rightsquigarrow_1$ in $\rightsquigarrow_\infty$).

Proof. By induction on t. The first map $\cdot_{-1}\overline{r}$ is clearly an embedding since the domain is empty. Now consider Diagram 2 defining $a_0 \rightsquigarrow_{t+2} a$. By inductive hypothesis (and the simple fact that a map on sigma-types is an embedding if it is fibrewise an embedding), the top map is an embedding, so $\iota_{t,a_0,a}$ is also an embedding by Theorem 7. It remains to show that $\cdot_{t+1}\overline{r}$ is also an embedding. To this end, suppose given $a : A$, $b : B$, $r : R(a,b)$, and $q, q' : a_0 \rightsquigarrow_{t+1} b$. Now by Theorem 7 and

[3]We abuse notation by passing freely between the descriptions of spans as diagrams and as relations.
[4]Explicitly, the map $R \to A$ is n-truncated if for each $a : A$, the type $(b : B) \times R(a,b)$ is n-truncated.

using the assumption that $(b' : B) \times R(a, b')$ is 0-truncated, the equality type $q \cdot_{t+1} \overline{r} = q' \cdot_{t+1} \overline{r}$ is given by the pushout

$$\begin{tikzcd}
(p : a_0 \rightsquigarrow_t a) \times (q = p \cdot_t r) \times (q = q') \arrow[r] \arrow[d] & (p : a_0 \rightsquigarrow_t a) \times (q = p \cdot_t r) \times (q' = p \cdot_t r) \arrow[d] \\
q = q' \arrow[r] & q \cdot_{t+1} \overline{r} = q' \cdot_{t+1} \overline{r} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

and our goal is to show that the bottom map is an equivalence. It is direct that the top map is an equivalence, since it is given by the evident equivalence $(q = q') \simeq (q' = f_{t,r}(p))$ for $q = f_{t,r}(p)$. This finishes the proof. □

Theorem 9. Suppose given a span A, B, R such that both maps $R \to A$ and $R \to B$ are 0-truncated, and such that A and B are both n-truncated with $n \geq 1$. Then the pushout $A +_R B$ is again n-truncated.

Proof. We have to show that $A +_R B$ has $(n-1)$-truncated loop spaces. Since to be truncated is a proposition, it suffices to consider $\mathsf{inl}(a) = \mathsf{inl}(a)$ and $\mathsf{inr}(b) = \mathsf{inr}(b)$. By symmetry it suffices to consider $\mathsf{inl}(a) = \mathsf{inl}(a)$. Since this is a homogeneous type, it suffices to show that its loop space at $\mathsf{refl}_{\mathsf{inl}(a)}$ is $(n-2)$-truncated. By Lemma 8, the action on paths $(a = a) \to (\mathsf{inl}(a) = \mathsf{inl}(a))$ is an embedding, and hence an equivalence on loop spaces. Since $a = a$ is $(n-1)$-truncated, we are done. □

Example 10. Given a span A, B, R where A and B are 1-types and R is a 0-type, it is automatic that $R \to A$ and $R \to B$ are 0-truncated. Thus $A +_R B$ is 1-truncated. Moreover the cartesian gap map in the pushout square is an embedding, corresponding to the embedding from $\rightsquigarrow_1$ to $\rightsquigarrow_\infty$.

Example 11. If X is a set, then the suspension $\Sigma X :\equiv 1 +_X 1$ is 1-truncated. In particular the free higher group on A is a set, namely $\Omega\Sigma(A+1)$. Moreover, the unit map $A + 1 \hookrightarrow \Omega\Sigma(A+1)$ is an embedding (corresponding to the cartesian gap map). This shows that a set embeds in its free group.

Example 12. For I a set and $X : I \to \mathcal{U}_{\mathsf{pt}}$ a family of pointed 1-types, the wedge $\bigvee_{i:I} X(i)$ is again a 1-type. To see this one considers the following diagram.

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & 1 \arrow[d] \\
(i : I) \times X(i) \arrow[r] & \bigvee_{i:I} X(i) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Moreover, the map $\bigvee_{i:I} \Omega X(i) \to \Omega\left(\bigvee_{i:I} X(i)\right)$ is an embedding, corresponding to the embedding from $* \rightsquigarrow_2 *$ to $* \rightsquigarrow_\infty *$, where $* : 1$. Note that if $X(i)$ are connected, then the wedge corresponds to a coproduct of groups, under the correspondence between groups and pointed connected 1-types.

4. Connectivity

We now look more closely at the maps in Diagram 1. Our results closely correspond to the connectivity results in Section 7 of [3]. For example, we will see that in good situations, the connectivity of the maps in the sequence increases linearly. We will also be able to prove a general form of Blakers–Massey [1], similar to the proof of the Freudenthal suspension theorem in [3]. The results in this section supersede those in the previous section.

We consider a fixed span A, B, R as before, with $a_0 : A$.

Lemma 13. Let $\mathcal{L}$ be the class of connected maps of some modality. Suppose that for some even t and all a, b, $r : R(a,b)$, the map $\cdot_t r$ is in $\mathcal{L}$. Then so is every map after $\cdot_t r$ in Diagram 1. The same holds if we instead consider odd t and $\cdot_t \overline{r}$.

Proof. The same as the proof of Lemma 3, using that $\mathcal{L}$ is closed under pushouts, transfinite composition, and satisfies the appropriate 2-out-of-3 property. ☐

It will be useful to have a more refined description of $\cdot_{t+1}\overline{r}$ than what is afforded by 2-out-of-3. We first recall the following lemma about composition of cogap maps. As in [1], we say that the cogap map of a square is the induced map from the pushout to the bottom right object.

Lemma 14. Consider a commutative rectangle as below.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd}$$

The cogap map is functorial in the sense that the cogap map of the whole square is a composition of a pushout of the cogap map of the left square, followed by the cogap map of the right square.

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
A \arrow[rr] \arrow[ddd] & & B \arrow[rrr] \arrow[dl] \arrow[ddd] & & & C \arrow[dll] \arrow[ddl] \arrow[ddd] \\
& P \arrow[rr] \arrow[ddr] & & Q \arrow[dr] & & \\
& & & & R \arrow[dr] & \\
X \arrow[uur] \arrow[rr] & & Y \arrow[uur] \arrow[urr] \arrow[rrr] & & & Z
\end{tikzcd}$$

Here $ABXP$, $ACXQ$, and $BCYR$ are pushout squares. The cogap map of the whole square is $Q \to Z$. It is the composite of $Q \to R$ and the cogap map $R \to Z$ of the right square. Thus it remains to show that $PQYR$ is a pushout square. By pushout pasting, it suffices to show that $BCPQ$ is a pushout square. This holds by pushout pasting applied to the squares defining P and Q. ☐

Now consider the following diagram, for given $a_0, a : A$, $b_0 : B$ and $r_0 : R(a, b_0)$.

$$\begin{tikzcd}
a_0 \rightsquigarrow_t a \arrow[r, "\cdot_t r_0"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & a_0 \rightsquigarrow_{t+1} b_0 \arrow[d] \\
(b : B) \times R(a,b) \times a_0 \rightsquigarrow_t a \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & (b : B) \times R(a,b) \times a_0 \rightsquigarrow_{t+1} b \arrow[d] \\
a_0 \rightsquigarrow_t a \arrow[r] & a_0 \rightsquigarrow_{t+2} a
\end{tikzcd} \tag{3}$$

In this diagram, the top square is a pullback and the bottom square is a pushout. The left vertical composite is the identity. The right vertical composite is $\cdot_{t+1}\overline{r}_0$.

Lemma 15. Let f be the cogap map of the top square above. Then $\cdot_{t+1}\overline{r}_0$ is a pushout of f.

Proof. Since the left composite is an equivalence, the right composite is the cogap map of the big square. By Lemma 14, this is the composite of a pushout of f followed by the cogap map of the bottom square. The bottom square is a pushout by definition, so its cogap map is an equivalence. □

Lemma 16. The cogap map in a pullback square is given by fibrewise join. That is, given a pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

the fibre of the cogap map over $w : W$ is the join of the fibre of $Z \to W$ over w and the fibre of $Y \to W$ over w.

Proof. By the definition of the join and the fact the pushouts commute with pullback, in this case along $w : 1 \to W$. □

This gives an explicit description of f. Say we are interested in the fibre of f over (b, r, p) where $b : B$, $r : R(a, b)$, and $p : a_0 \rightsquigarrow_{t+1} b$. Then, after some simplifications, Lemma 16 says that this is the join of the fibre of $\cdot_t r$ over p and $(b, r) = (b_0, r_0)$. Note that the latter is a fibre of the diagonal of $R \to A$. Explicitly, the diagonal of $R \to A$ is the map

$$(a : A) \times (b : B) \times R(a, b) \to (a : A) \times (b\, b' : B) \times R(a, b) \times R(a, b'),$$

and our equality type $(b, r) = (b_0, r_0)$ is the fibre over (a, b_0, b, r_0, r).

This can be used to give another proof of Theorem 7, using that the join of any type with a contractible type is contractible, and that the diagonal of an embedding is an equivalence. It also gives another proof of Lemma 8, that $\cdot_{t+1}\overline{r}_0$ is an embedding when the maps $R \to A$ and $R \to B$ are both 0-truncated: $\cdot_{t+1}\overline{r}_0$ is a pushout of f, and f is an embedding because its fibres are joins of propositions (using an inductive hypothesis).

We will use the well-known join connectivity lemma without proof.

Lemma 17. The join $X * Y$ of an m-connected type X with an n-connected type Y is $(m+n+2)$-connected.

Theorem 18. Say given $m, n : \mathbb{N}$ and a span A, B, R where the diagonals of $R \to A$ and $R \to B$ are m- and n-connected, respectively. Then for any $a_0, a : A$, $b : B$, $r : R(a, b)$, and $k : \mathbb{N}$, the map $\cdot_{2k-1}\overline{r} : (a_0 \rightsquigarrow_{2k-1} b) \to (a_0 \rightsquigarrow_{2k} a)$ is $(k(m+n+4)-2)$-connected, and the map $\cdot_{2k} r : (a_0 \rightsquigarrow_{2k} a) \to (a_0 \rightsquigarrow_{2k+1} b)$ is $(k(m+n+4)+n)$-connected.

It follows by Lemma 13 that $(a_0 \rightsquigarrow_{2k-1} b) \to (a_0 \rightsquigarrow_{\infty} b)$ is also $(k(m+n+4)-2)$-connected, and that $(a_0 \rightsquigarrow_{2k} a) \to (a_0 \rightsquigarrow_{\infty} a)$ is $(k(m+n+4)+n)$-connected. In particular, taking $k = 1$ in the first statement we see that the cartesian gap map $R(a, b) \to (\mathsf{inl}(a) = \mathsf{inr}(b))$ is $(n+m+2)$-connected. This is the classical Blakers–Massey theorem. The theorem works also for joins of general modalities, as in [1], but we state it for m-connectivity for concreteness.

Proof. By induction on k. The first statement is vacuous for $k = 0$, since every map is (-2)-connected. For a given k, we derive the second statement from the first. By a version of the above discussion with the roles of A and B switched, the map $\cdot_{2k} r$ is the pushout of a map f whose fibres are joins of fibres of $\cdot_{2k-1}\overline{r}'$ and fibres of the diagonal of $R \to B$. Since connected maps are closed under pushout, it suffices to consider f instead of $\cdot_{2k} r$. We use the description from above of the fibres of f as joins. By assumption, the fibres of $\cdot_{2k-1}\overline{r}'$ are $(k(m+n+4)-2)$-connected. Moreover the fibres of the diagonal of $R \to B$ are n-connected. The desired result follows by join connectivity, since $(k(m+n+4)-2)+n+2 = k(m+n+4)+n$.

In the same way, one shows that the second statement for a given k gives the first statement for $k+1$, using $k(m+n+4)+n+m+2=(k+1)(m+n+4)-2$. □

5. Historical discussion

Brunerie formalised a proof that $\Omega\Sigma(A+1)$ is a set if A has decidable equality in [2], using reduced words as normal forms. This was later posed as Exercise 8.2 in [9], with the extension to arbitrary sets mentioned as an open problem. Kraus and Altenkirch [5] expressed the problem as describing the free higher group on a set. In this way one can understand it as a coherence problem for higher groups, analogous to the coherence theorems for monoidal categories and bicategories. They also showed that the 1-truncation of the free higher group is 0-truncated, using ideas about confluent rewriting, similar to the constructive proof in [7, Chapter X] that a set embeds in its free group. Christian Sattler proposed to understand the free higher group on a set by observing that any set is a filtered colimit of finite sets, and using that suspension and loop spaces commute with filtered colimits. This gives, for every *external* natural number k, a proof in HoTT that the k-truncation of the free higher group on a set is 0-truncated. The issue with obtaining the full result is that we do not have any way of understanding colimits over general diagrams in HoTT. The lack of concrete progress toward the full result until now had led several experts to believe that HoTT would not be strong enough to prove that the free higher group on a set is a set.

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