CONFLUENT COLIMITS COMMUTE WITH PULLBACKS, GIVEN DESCENT

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Abstract. We present a simple, model-independent proof that filtered colimits commute with finite limits in any $(\infty, 1)$-category with finite limits and filtered colimits that satisfy descent. This includes the $(\infty, 1)$-category of ∞-groupoids as well as any other ∞-topos. We also show that a larger class of colimits, called confluent colimits, commute with pullbacks under similar assumptions. This simplifies results of Lurie. Our proof is extracted from a result in homotopy type theory due to Sojakova, van Doorn and Rijke.

1. Introduction

A fundamental result in 1-category theory says that filtered colimits in Set commute with finite limits. In his foundational work on higher topos theory [1], Lurie established the corresponding result in higher category theory: filtered colimits in the $(\infty, 1)$-category $\mathcal{S}$ of spaces (or any other ∞-topos) commute with finite limits. A somewhat simpler proof appears in Kerodon [2, Tag 05XR].

In this note we present a yet simpler proof that filtered colimits commute with finite limits in $\mathcal{S}$. Unlike previous proofs, ours is model-independent in the sense that it can readily be applied to any reasonable notions of '$(\infty, 1)$-category' and 'space', including in the internal logic of an $(\infty, 1)$-topos. We rely only on the characterisation of cofinal functors, the theory of pointwise Kan extensions, straightening/unstraightening for left fibrations, and descent in $\mathcal{S}$. In fact our proof applies not only to $\mathcal{S}$ but to any $(\infty, 1)$-category with descent.

By definition, general finite limits can be expressed in terms of pullbacks and terminal objects. Also by definition, filtered colimits commute with terminal objects, and so the non-trivial claim is that filtered colimits commute with pullbacks. Here it is natural to consider not only filtered colimits, but colimits indexed by a more general class of categories which we dub *confluent*. We show that confluent colimits commute with pullbacks, meaning that pullbacks form a sound doctrine. This fact was shown independently by Rezk [3] and the first author in earlier work. It is a phenomenon specific to higher category theory: pullbacks do not form a sound doctrine in 1-category theory.

We will present two versions of the argument. The first is expressed in terms of fibrations of categories, i.e. it is 'unstraightened', and applies to $\mathcal{S}$ but not general categories. The second version is 'straightened' and applies to any $(\infty, 1)$-category with descent. We emphasize that these two proofs can be read independently of each other. They follow the same pattern and are based on the same core idea.

At a non-rigorous level, the idea can be explained as follows. For J confluent (see Definition 1), we would like to establish an equivalence of the following form, given a cospan of diagrams $A \to C \leftarrow B$.

$$\operatorname*{colim}_{x:J} A(x) \times_{C(x)} B(x) \simeq \operatorname{colim} A \times_{\operatorname{colim} C} \operatorname{colim} B$$

We first establish the following intermediate result involving a map of diagrams $B \to C$ and an object $x : J$. This result crucially relies on descent.

$$C(x) \times_{\operatorname{colim} C} \operatorname{colim} B \simeq \operatorname*{colim}_{y:J,\ f:x\to y} C(x) \times_{C(y)} B(y)$$

Given this, we compute as follows.

$$
\begin{aligned}
&\operatorname{colim} A \times_{\operatorname{colim} C} \operatorname{colim} B \\
&\simeq \operatorname*{colim}_{x:J} (A(x) \times_{\operatorname{colim} C} \operatorname{colim} B) && \text{by universality of colimits} \\
&\simeq \operatorname*{colim}_{x:J} \big(A(x) \times_{C(x)} C(x) \times_{\operatorname{colim} C} \operatorname{colim} B\big) && \text{by pullback pasting} \\
&\simeq \operatorname*{colim}_{x:J} \left(A(x) \times_{C(x)} \operatorname*{colim}_{y:J,\ f:x\to y} C(x) \times_{C(y)} B(y)\right) && \text{by intermediate result above} \\
&\simeq \operatorname*{colim}_{x:J} \operatorname*{colim}_{y:J,\ f:x\to y} A(x) \times_{C(y)} B(y) && \text{by universality of colimits} \\
&\simeq \operatorname*{colim}_{x:J,\ y:J,\ f:x\to y} A(x) \times_{C(y)} B(y) && \text{by formula for iterated colimit} \\
&\simeq \operatorname*{colim}_{x:J} A(x) \times_{C(x)} B(x) && \text{by cofinality}
\end{aligned}
$$

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2. Preliminaries

We consider higher categorical and homotopy-invariant notions by default. We take 'category' to mean $(\infty, 1)$-category. We write $\mathcal{S}$ for the category of spaces and Cat for the category of small categories. Given categories $\mathcal{C}$, $\mathcal{D}$, we write $\mathcal{D}^{\mathcal{C}}$ for the category of functors from $\mathcal{C}$ to $\mathcal{D}$. We write $[1]$ for the category freely generated by objects $0, 1 : [1]$ and a morphism $0 \to 1$. We write $\mathcal{C}^{\to}$ for the category $\mathcal{C}^{[1]}$ of arrows in $\mathcal{C}$. We say a functor $p : \mathcal{D} \to \mathcal{C}$ is a left (resp. right) fibration if it is right orthogonal to the left endpoint inclusion $\{0\} \to [1]$ (resp. the right endpoint inclusion $\{1\} \to [1]$). Explicitly, this means that the map $\mathcal{D}^{\to} \to \mathcal{D} \times_{\mathcal{C}} \mathcal{C}^{\to}$ is an equivalence, where $\mathcal{C}^{\to} \to \mathcal{C}$ selects the domain of a morphism. We say that p is a Kan fibration if it is both a left fibration and a right fibration.

We denote the left adjoint to the forgetful functor $\mathcal{S} \to \mathsf{Cat}$ by $|-| : \mathsf{Cat} \to \mathcal{S}$ and call it localisation. For any category $\mathcal{C}$, the forgetful functor $\mathsf{Lfib}(\mathcal{C}) \to \mathsf{Cat}/\mathcal{C}$ from left fibrations over $\mathcal{C}$ to categories over $\mathcal{C}$ admits a left adjoint which can be computed as follows. Given a functor $p : \mathcal{D} \to \mathcal{C}$, the free cocartesian fibration on p is given by $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}^{\to}$, viewed as a category over $\mathcal{C}$ via the codomain projection $\mathcal{C}^{\to} \to \mathcal{C}$. The free left fibration on p is the fibrewise localisation of the free cocartesian fibration, so that the fibre over $c : \mathcal{C}$ is given by $|\mathcal{D} \downarrow c|$ where $\mathcal{D} \downarrow c$ is the category $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}^{\to} \times_{\mathcal{C}} \{c\}$. Dually, the free right fibration on p has fibres given by $|c \downarrow \mathcal{D}|$.

We say a functor $p : A \to B$ is cofinal if its reflection to $\mathsf{Rfib}(B)$ is an equivalence. Explicitly, this means that for all $b : B$, the space $|b \downarrow A|$ is contractible. Cofinal maps and right fibrations form a factorisation system on Cat, where the factorisation of $p : \mathcal{D} \to \mathcal{C}$ is given by the free right fibration on p described above. Right fibrations are stable under base change along arbitrary functors, and cofinal functors are stable under base change along *left* fibrations. So the factorisation of a map into a cofinal functor followed by a right fibration is stable under base change along left fibrations.

We write Λ^2_0 for the walking span, i.e. the category $1 \leftarrow 0 \to 2$. Thus a functor $s : \Lambda^2_0 \to \mathcal{C}$ is given by a span $s_1 \leftarrow s_0 \to s_2$ in $\mathcal{C}$.

Descent. In the fibrational perspective, we will use descent in the following form. By straightening-unstraightening, Kan fibrations $\mathcal{D} \to \mathcal{C}$ are classified by functors $\mathcal{C} \to \mathcal{S}^{\simeq}$ into the groupoidal core of $\mathcal{S}$. These in turn correspond to maps $|\mathcal{C}| \to \mathcal{S}$, which unstraighten to spaces $X \to |\mathcal{C}|$ over the localisation of $\mathcal{C}$. Given a map of spaces $X \to |\mathcal{C}|$, the corresponding Kan fibration is the pullback $X \times_{|\mathcal{C}|} \mathcal{C}$. Given an arbitrary functor $p : \mathcal{D} \to \mathcal{C}$, the free Kan fibration on p corresponds to the map of spaces $|p| : |\mathcal{D}| \to |\mathcal{C}|$.

3. The fibrational perspective

Our aim in this section is to give a fibrational proof that confluent colimits commute with pullback.

Definition 1. *A category J is* confluent *if the diagonal functor $\Delta : J \to J^{\Lambda^2_0}$ from J into the category of spans in J is cofinal.*

Explicitly, this means that for any span $s : \Lambda^2_0 \to J$, the category $s \downarrow J$ of cocones under s has contractible localisation.

Lemma 2. *A category J is confluent if and only if for every morphism $f : x \to y$ in J, the functor $f^! : y \downarrow J \to x \downarrow J$ given by composition with f is cofinal.*

Proof. We have that $f^!$ is cofinal if and only if, for every object $g : x \to x'$ of $x \downarrow J$, the category $(x', g) \downarrow (y \downarrow J)$ has contractible localisation. This is equivalent to the category of cocones under the span $y \xleftarrow{f} x \xrightarrow{g} x'$. □

Lemma 3. *If J is confluent and $p : E \to J$ is a left fibration, then E is confluent.*

Proof. This can be deduced from the fact that Λ^2_0 has an initial object. But we instead argue from Lemma 2.

Given a morphism $g : a \to b$ in E, we have to show that $g^! : b \downarrow E \to a \downarrow E$ is cofinal. Consider the following commutative square of categories.

$$\begin{array}{ccc} b \downarrow E & \xrightarrow{g^!} & a \downarrow E \\ \downarrow & & \downarrow \\ p(b) \downarrow J & \xrightarrow[p(g)^!]{} & p(a) \downarrow J \end{array}$$

Since p is a left fibration, the vertical maps are equivalences. Since J is confluent, the bottom map is cofinal. So the top map is also cofinal. □

Lemma 4. *Let J be a confluent category and $p : E \to J$ a left fibration. Then the free right fibration on p is a left fibration.*

This means that the free right fibration on a left fibration is also the free Kan fibration. In this way we get a remarkably simple description of free Kan fibrations.

Proof. Let $q : \overline{E} \to J$ denote the free right fibration on p, so that the goal is to show that q is a Kan fibration. That is, we have to show that for a morphism $f : x \to y$ in J, the map $\overline{E}(y) \to \overline{E}(x)$ on fibres is an equivalence of spaces.

Now $\overline{E}(x)$ is given by $\operatorname{colim} E|_{x\downarrow J}$, where $E|_{x\downarrow J} : x \downarrow J \to \mathcal{S}$ denotes the restriction of $E : J \to \mathcal{S}$ along $x \downarrow J \to J$. The map $\overline{E}(y) \to \overline{E}(x)$ corresponds to the map $\operatorname{colim} E|_{y\downarrow J} \to \operatorname{colim} E|_{x\downarrow J}$ induced by restricting the shape of the colimit along $f^! : y \downarrow J \to x \downarrow J$. This is an equivalence by Lemma 2. □

Lemma 5. *Let $F : J' \to J$ be a left fibration with J confluent. Then base change along F preserves reflection of left fibrations to Kan fibrations.*

Explicitly, this means that if $p : E \to J$ is a left fibration and $q : \overline{E} \to J$ is the free Kan fibration on p, then the induced map $J' \times_J E \to J' \times_J \overline{E}$ over J' exhibits $J' \times_J \overline{E} \to J'$ as the free Kan fibration on $J' \times_J E \to J'$.

Proof. By Lemma 3, J' is also confluent. By Lemma 4, we can consider the reflection from left fibrations to right fibrations instead of to Kan fibrations. Indeed base change along any left fibration preserves reflection (from arbitrary functors) to right fibrations. □

We are now ready to prove the main result.

Theorem 6. *In the category $\mathcal{S}$ of spaces, confluent colimits commute with pullbacks.*

Proof. Let J be a confluent category and consider a pullback square of functors $J \to \mathcal{S}$. By straightening, these correspond to left fibrations over J:

$$\begin{array}{ccc} Y' & \longrightarrow & Y' \\ \big\downarrow & \lrcorner & \big\downarrow \\ X' & \longrightarrow & X \end{array}$$

We have to show that the induced square of localisations is a pullback:

$$\begin{array}{ccc} |Y'| & \longrightarrow & |Y'| \\ \big\downarrow & & \big\downarrow \\ |X'| & \longrightarrow & |X| \end{array}$$

By descent, this equivalently means that the following square is a pullback, where $\overline{Y} \to X$ and $\overline{Y'} \to X'$ are the free Kan fibrations on $Y \to X$ and $Y \to X$.

$$\begin{array}{ccc} \overline{Y'} & \longrightarrow & \overline{Y} \\ \big\downarrow & & \big\downarrow \\ X' & \longrightarrow & X \end{array}$$

We know that X is confluent, by Lemma 3. So the above square is a pullback by Lemma 5. $\square$

4. Unstraightened perspective

Now suppose $\mathcal{C}$ is an arbitrary category and J is confluent. We assume throughout that $\mathcal{C}$ has pullbacks, J-colimits that satisfy descent, and universal colimits in general. We would like to show that J-colimits commute with pullbacks in $\mathcal{C}$.

Concretely, asking that J-colimits in $\mathcal{C}$ satisfy descent means the following. Given a functor $A : J \to \mathcal{C}$, we have an adjunction between the slice categories $\mathcal{C}^J/A$ and $\mathcal{C}/\operatorname{colim} A$. The left adjoint $\mathcal{C}^J/A \to \mathcal{C}/\operatorname{colim} A$ sends an object $B \to A$ over A to $\operatorname{colim} B \to \operatorname{colim} A$. The right adjoint sends an object $Y \to \operatorname{colim} A$ to the pullback $A \times_{\Delta \operatorname{colim} A} \Delta Y$ viewed as an object over A via the first projection, where $\Delta : \mathcal{C} \to \mathcal{C}^J$ is the diagonal functor.

The colimit of A satisfies descent if this adjunction defines an equivalence between $\mathcal{C}/\operatorname{colim} A$ and the full subcategory of $\mathcal{C}^J/A$ spanned by those objects that are *equifibred* over A. Given $p : B \to A$, we say that B is equifibred over A if for every morphism $f : x \to y$, the following naturality square is cartesian.

$$\begin{array}{ccc} B(x) & \longrightarrow & B(y) \\ \big\downarrow & \lrcorner & \big\downarrow \\ A(x) & \longrightarrow & A(y) \end{array}$$

We will refer to the reflection from $\mathcal{C}^J/A$ to its full subcategory of equifibred objects as *equifibred replacement*. We claim that equifibred replacement has the following explicit description. Suppose that $p : B \to A$ is some object over A, and consider the functor $H : J^{\to} \to \mathcal{C}$ given by the pullback $H \coloneqq (A \circ \mathsf{dom}) \times_{A \circ \mathsf{cod}} (B \circ \mathsf{cod})$. Our aim is to show that the left Kan extension $\widetilde{B} : J \to \mathcal{C}$ of H along $\mathsf{dom} : J^{\to} \to J$ is the equifibred replacement of $p : B \to A$.

We first give a description of $\widetilde{B}$. By the theory of pointwise Kan extensions, given $x : J$ we have that $\widetilde{B}(x)$ is the colimit in $\mathcal{C}$ of the restriction of H to $J^{\to} \downarrow x$. Explicitly, $J^{\to} \downarrow x$ is the category of spans $y \leftarrow z \to x$ in J where the endpoint x is fixed. Consider the functor $x \downarrow J \to J^{\to} \downarrow x$ which sends a morphism $f : x \to y$ to the span $y \xleftarrow{f} x \xrightarrow{\mathsf{id}} x$.

Lemma 7. *The functor $x \downarrow J \to J^{\to} \downarrow x$ defined above is cofinal.*

Proof. Let $y \xleftarrow{f} z \xrightarrow{g} x$ be an arbitrary object of $J^{\to} \downarrow x$. We need to show that the category $(y \xleftarrow{f} z \xrightarrow{g} x) \downarrow (x \downarrow J)$ has contractible localisation. This is the category consisting of $c : \mathcal{C}$ with $h_x : x \to c$ and a morphism of spans from $y \xleftarrow{f} z \xrightarrow{g} x$ to $c \xleftarrow{h_x} x \xrightarrow{\mathsf{id}} x$, whose component map $x \to x$ is the identity. The category of maps $g' : z \to x$ with a witness that $g' \circ \mathsf{id}_x \simeq g \circ \mathsf{id}_x$ is trivial, so the only data in the morphism of spans is a map $h_y : y \to c$ together with a witness that $h_y \circ f \simeq h_x \circ g$. Thus $(y \xleftarrow{f} z \xrightarrow{g} x) \downarrow (x \downarrow J)$ is equivalent to the category of cocones under $y \xleftarrow{f} z \xrightarrow{g} x$, which has contractible localisation since J is confluent. □

Thus $\widetilde{B}(x)$ is the colimit of the restriction of H to $x \downarrow J$. This restriction $H|_{x\downarrow J} : x \downarrow J \to \mathcal{C}$ is a pullback $(\Delta A(x)) \times_{A \circ \mathsf{cod}} (B \circ \mathsf{cod})$.

The universal property of the left Kan extension $\widetilde{B}$ says that it is freely generated by a natural transformation $\alpha : H \to \widetilde{B} \circ \mathsf{dom}$. Whiskering α with $\Delta : J \to J^{\to}$, we get $\alpha_\Delta : H \circ \Delta \to \widetilde{B} \circ \mathsf{dom} \circ \Delta$. Since $H \circ \Delta \simeq B$ and $\mathsf{dom} \circ \Delta \simeq \mathsf{id}$, we get a natural transformation $\iota : B \to \widetilde{B}$. The first projection gives a natural transformation $\pi_1 : H \to A \circ \mathsf{dom}$, so by the universal property of $\widetilde{B}$ we have $q : \widetilde{B} \to A$ with $\pi_1 = (q_{\mathsf{dom}}) \circ \alpha$. It is straightforward to check that $B \xrightarrow{\iota} \widetilde{B} \xrightarrow{q} A$ is a factorisation of the original natural transformation $p : B \to A$.

Lemma 8. *$\widetilde{B}$ is equifibred over A.*

Proof. So far we have seen a description of the action of $\widetilde{B}$ on objects $x : J$. We now also need to describe its action on morphisms $f : x \to x'$. This is not entirely immediate, since the shape $x \downarrow J$ of the colimit describing $\widetilde{B}(x)$ is contravariant in x. The action on morphisms is naturally described in terms of a *span*. We refer to the following diagram.

$$\begin{array}{ccccc}
\operatorname{colim}_{x\downarrow J} H|_{x\downarrow J} & \xleftarrow{\sim} & \operatorname{colim}_{x'\downarrow J} H|_{x\downarrow J} \circ f^! & \longrightarrow & \operatorname{colim}_{x'\downarrow J} H|_{x'\downarrow J} \\
& \searrow^{\sim} & \downarrow & & \downarrow{\scriptstyle\sim} \\
& & \widetilde{B}(x) & \longrightarrow & \widetilde{B}(x') \\
& & \downarrow & & \downarrow \\
& & A(x) & \longrightarrow & A(x')
\end{array}$$

The top left morphism is induced by restricting the colimit shape along $f^!$, and the top right morphism by a transformation of diagrams of the same shape $x' \downarrow J$. The top left morphism anyway ends up being an equivalence since $f^!$ is cofinal by Lemma 2. Thus the upper middle morphism is also an equivalence.

We would like to show that the bottom square is cartesian; equivalently this means showing that the outer (right) square is cartesian. Since we assume $\mathcal{C}$ has universal colimits, it suffices to show that for every object $g : x' \to y$ of $x' \downarrow J$, the following square is cartesian.

$$\begin{array}{ccc}
(H|_{x\downarrow J} \circ f^!)(y, g) & \longrightarrow & H|_{x'\downarrow J}(y, g) \\
\downarrow & & \downarrow \\
A(x) & \longrightarrow & A(x')
\end{array}$$

The square above can be seen to be equivalent to the left square in the following diagram. So it is cartesian by pasting.

$$\begin{array}{ccccc} A(x) \times_{A(y)} B(y) & \longrightarrow & A(x') \times_{A(y)} B(y) & \longrightarrow & B(y) \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ A(x) & \longrightarrow & A(x') & \longrightarrow & A(y) \end{array}$$

□

Lemma 9. *The natural transformation $\iota : B \to \widetilde{B}$ induces an equivalence on colimits.*

Proof. By composition of left Kan extensions, the colimit of $\widetilde{B}$ is equivalent to the colimit of H. This is in turn equivalent to the colimit of $H \circ \Delta : J \to \mathcal{C}$, since $\Delta : J \to J^{\to}$ is cofinal (with left adjoint $\mathsf{dom} : J^{\to} \to J$). Now note that $H \circ \Delta \simeq B$. □

Corollary 10. *The factorisation $B \xrightarrow{\iota} \widetilde{B} \xrightarrow{q} A$ of $p : B \to A$ exhibits $\widetilde{B}$ as the equifibred replacement of B.*

Proof. By Lemma 8 and Lemma 9, using descent. □

Lemma 11. *Equivariant replacement is stable under base change along $A' \to A$.*

Proof. We start from a cartesian square in $\mathcal{C}^J$:

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ A' & \longrightarrow & A \end{array}$$

This factorises into the following diagram involving equifibred replacement, since equifibred natural transformations are at least stable under pullback.

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \downarrow & & \downarrow \\ \widetilde{B}' & \longrightarrow & \widetilde{B} \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

Our goal is to show that the bottom map is cartesian. Given an object $x : J$, we have to show that $\widetilde{B}'(x) \to A'(x)$ is exhibited as a pullback of $\widetilde{B}(x) \to A(x)$. That is, we have to show that the following square is cartesian.

$$\begin{array}{ccc} \operatorname{colim}_{x\downarrow J} \Delta A'(x) \times_{A' \circ \mathsf{cod}} (B' \circ \mathsf{cod}) & \longrightarrow & \operatorname{colim}_{x\downarrow J} \Delta A(x) \times_{A \circ \mathsf{cod}} (B \circ \mathsf{cod}) \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

This follows from pullback pasting and universality of colimits. □

Theorem 12. *Let J be a confluent category and let $\mathcal{C}$ be a category with pullbacks. Suppose that $\mathcal{C}$ has J-colimits that satisfy descent. Suppose also that $\mathcal{C}$ has universal $x \downarrow J$-shaped colimits for all objects $x : J$. Then J-colimits commute with pullbacks in $\mathcal{C}$.*

Proof. Again we start from a cartesian square in $\mathcal{C}^J$:

$$\begin{array}{ccc} B' & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ A' & \longrightarrow & A \end{array}$$

We would like to show that the colimit functor $\mathcal{C}^J \to \mathcal{C}$ sends this to a cartesian square in $\mathcal{C}$:

$$\begin{array}{ccc} \operatorname{colim} B' & \longrightarrow & \operatorname{colim} B \\ \downarrow & & \downarrow \\ \operatorname{colim} A' & \longrightarrow & \operatorname{colim} A \end{array}$$

By descent, this equivalently means that the induced square involving equifibred replacements is cartesian:

$$\begin{array}{ccc} \widetilde{B}' & \longrightarrow & \widetilde{B} \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

Indeed this holds by Lemma 11. □

Corollary 13. *Let J be a confluent category, and let $\mathcal{C}$ be a category with pullbacks, J-shaped colimits that satisfy descent, and universal filtered colimits. Then J-shaped colimits commute with pullbacks in $\mathcal{C}$.*

Proof. By Theorem 12, it suffices to show that if J is confluent, then $x \downarrow J$ is filtered for any object $x : J$. Since $\mathtt{cod} : x \downarrow J \to J$ is a left fibration and J is confluent, $x \downarrow J$ is confluent. So $x \downarrow J$-shaped colimits commute with pullbacks in spaces. We also know that $x \downarrow J$ has contractible localisation, since it has an initial object. So $x \downarrow J$-shaped colimits commute with arbitrary finite limits. A direct argument shows that this means $x \downarrow J$ is filtered [3]. □

As shown by Rezk [3], we have the following relationship between filtered and confluent categories. On one hand, by the above we know that a category is filtered if and only if it is confluent and has contractible localisation. Examples of categories that are confluent but usually not filtered include groupoids (by Lemma 3 with $J = 1$). In some sense this is the only source of examples: given a category J, we can view it as a groupoid-indexed family of categories by straightening the functor $J \to |J|$. Rezk shows that J is confluent if and only if all the fibres of this functor $J \to |J|$ are filtered.

5. Discussion

The novel feature of our proof compared to earlier $(\infty, 1)$-categorical arguments is the idea of using descent to describe equifibred replacement, or in the fibrational perspective, reflection to Kan fibrations. This idea is ubiquitous in HoTT, where it goes by the name of *the encode-decode method*. In the case at hand, our description of equifibred replacement is an adaptation of the main result in Sojakova, van Doorn, and Rijke's treatment of sequential colimits in HoTT [4].

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