Unordered addition from biproducts*

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A fundamental fact about commutative monoids is that given a finite number of elements, their sum can be computed by adding the elements in any order. The naturals under addition form a typical example of a commutative monoid. A natural number can be understood as an isomorphism class of finite sets, and indeed the type FinSet of finite sets under coproduct forms a higher kind of commutative monoid – to be precise, a symmetric monoidal category. While $\mathbb{N}$ is a 0-type and FinSet is a 1-type, there is also a good generalisation of commutative monoids to general homotopy types, namely E_∞-spaces. The invertible counterpart to E_∞-spaces spaces are connective spectra. While (connective) spectra can easily be defined in homotopy type theory – in short, one considers a sequence of pointed types where each one is the loop space of the next – a definition of general E_∞-spaces in homotopy type theory remains elusive.

The problem is the same as with other higher algebraic notions: there is an infinite tower of coherences involved and we do not know how to state them all at once. What we can do is state some of them. In particular, the fundamental fact about commutative monoids mentioned above is expected to generalise to E_∞-spaces in the following way. We say a type I is finite if I merely is equivalent to a standard finite type $[n]$ for some natural n, called the cardinality of I. An unordered (n-)tuple of elements of X consists of a finite type I (of cardinality n) together with a map $I \to X$. Given an unordered tuple of elements of an E_∞-space, we expect to be able to define their sum. Moreover, we expect the usual rules for indexed sums to hold, such as $\sum_i \sum_j a_{ij} = \sum_{(i,j)} a_{ij}$. In short, we say that E_∞-spaces have unordered addition. For example, FinSet has unordered addition since given a tuple $J : I \to \mathsf{FinSet}$, the sigma-type $(i : I) \times J(i)$ is again a finite set.

While the notion of unordered addition is easy to express synthetically, it encapsulates a lot of information. For example, writing $\{a, b\}$ for the unordered pair given by the map $(a, b) : [2] \to A$, we have $\{a, b\} = \{b, a\}$ since there is an equivalence $[2] \simeq [2]$ swapping the two elements. Hence any addition operation defined on unordered pairs must be commutative in the naive sense that $a + b = b + a$. Moreover, the composite path $\{a, b\} = \{b, a\} = \{a, b\}$ is always reflexivity (essentially because the composite equivalence $[2] \simeq [2] \simeq [2]$ is the identity), meaning that the composite path $a + b = b + a = a + b$ is reflexivity. These are only two cases of an infinite tower of coherences.

Now if connective spectra are examples of E_∞-spaces, and E_∞-spaces have unordered addition, it should follow that (connective) spectra have unordered addition. The goal of this work is to prove this in homotopy type theory.

Let us first recall how addition is defined on a spectrum. First one defines general composition of paths, using path induction. This specialises to give composition in loop spaces. Since we have infinite loop spaces in mind, we call this operation addition of loops. By induction on n, one can then define the sum of an ordered n-tuple of loops. Since the underlying type of a spectrum is in particular a loop space, this defines ordered addition on (the underlying type of) a spectrum.

Unfortunately the above does not seem to help in defining unordered addition. Given an unordered n-tuple of elements of X, we cannot apply induction on n since we cannot pick an

*Under construction.

element of a general type I of cardinality $n \geq 2$. Moreover, there is an obstruction to defining unordered composition on a general loop space: general loop spaces might not be commutative. Any second loop space is commutative, by Eckmann–Hilton, but there is again an obstruction to defining unordered composition since the braiding induced by Eckmann–Hilton may fail to satisfy syllepsis.

Our solution to defining unordered addition in a spectrum is to take a step back and characterise addition by a defining property.[1] Given a spectrum X and a type I, we can form a new spectrum X^I which is the categorical product of I-many copies of X [4, Definition 5.4.1]. For a finite type I, unordered addition will be a map $X^I \to X$ of spectra, characterised in the following way. For each $i : I$, we get a map $\delta_i : X \to X^I$ whose jth component for $j : I$ is id_X if $i = j$ and 0 otherwise. Here we used that finite types have decidable equality, and that there is a 'zero morphism' from any spectrum to any other. Addition will be the unique morphism $\Sigma : X^I \to X$ with the structure that for each $i : I$ the composite $\Sigma \circ \delta_i : X \to X^I \to X$ is the identity on X. An upshot of this characterisation is that 'there is a unique morphism with the following structure' is a *proposition*, so to prove it we may assume I is purely equivalent to a standard finite type $[n]$. The statement is immediate for $n = 0$, and the following theorem deals with the case $n = 2$.

Theorem 1. *Given spectra A, B, C and morphisms $f : A \to C$, $g : B \to C$, the type of triples (h, p, q) where $h : A \times B \to C$[2], $p : f = h \circ (\mathsf{id}_A, 0)$ and $q : g = h \circ (0, \mathsf{id}_B)$ is contractible. In other words, there is a certain equivalence between $A \times B \to C$ and $(A \to C) \times (B \to C)$.*

Proof sketch. While a direct proof should be feasible, it is easy to get bogged down in path algebra. We suggest a somewhat indirect proof. One first shows that for any spectra X, Y, the type of morphisms $X \to Y$ is again a spectrum, since $(X \to Y) \simeq \Omega(X \to \Omega^{-1}Y)$. In particular, $A \times B \to C$ is a spectrum, and so has a well-defined addition. Hence given f and g, we obtain a map $f \circ \mathsf{fst} + g \circ \mathsf{snd} : A \times B \to C$. Verifying that this determines an equivalence between $A \times B \to C$ and $(A \to C) \times (B \to C)$ amounts to showing distributivity of composition together with the relation $(\mathsf{id}_A, 0) \circ \mathsf{fst} + (0, \mathsf{id}_B) \circ \mathsf{snd} = \mathsf{id}_{A \times B}$. This last relation can be understood as a simple fact about $\Omega(A \times B)$: an element of $\Omega(A \times B)$ is given by the sum of its component in $\Omega(A)$ and its component in $\Omega(B)$. □

In short, we say that the (wild) category of spectra has *biproducts*. A more unbiased way of saying this is that finite coproducts commute with finite products. This is a special case of fibre sequences coinciding with cofibre sequences, and of finite colimits commuting with finite limits, which is a defining feature of *stable* categories, of which the category of spectra is the prime example.

Theorem 2. *Given a spectrum X and a finite type I, the type of pairs (Σ, p), where $\Sigma : X^I \to X$ and $p : (i : I) \to \mathsf{id}_X = \Sigma \circ \delta_i$, is contractible.*

Proof sketch. More generally, we claim that in any wild higher category with associativity and unit laws, given a zero object and biproducts, we also have that finite products are coproducts in this way. The proof proceeds by a standard 1-categorical argument of reducing n-products to binary products, carefully keeping track of witnesses of commutativity along the way. This map Σ simply corresponds to the codiagonal map from a coproduct of copies of X back to X. □

[1] More precisely, this will generally be structure and not property.

[2] $A \times B$ denotes the product in the category of spectra, which as a sequence of pointed types is given component-wise by products of pointed types.

Thus spectra do have unordered addition. We define addition as a map of spectra, but it in particular gives an operation $\Sigma : X_0^I \to_{\mathrm{pt}} X_0$ on the underlying type X_0 of X. From its characterisation, we can also directly verify expected properties.

We now discuss a couple of applications. First we consider Eckmann-Hilton, which says that for any type A, the second loop space $\Omega^2 A$ has commutative addition. In general, $\Omega^2 A$ will not be a spectrum, and indeed it might not have unordered addition. However by the stabilisation theorem [1], the 0-truncation $\|\Omega^2 A\|_0$ is always a spectrum in a canonical way. Hence $\|\Omega^2 A\|_0$ has unordered addition. It can be seen that this addition is compatible with the ordered addition on $\Omega^2 A$. Thus for $a, b : \Omega^2 A$, we have $|a+b|_0 = |b+a|_0$, or equivalently, $\|a+b = b+a\|_{-1}$. This can be understood as a weak form of Eckmann–Hilton. Applying this result to the universal case of $a, b : \Omega^2(S^2 \vee S^2)$ and appealing to the existence property of homotopy type theory[3], one can recover the full strength of Eckmann–Hilton. More interestingly, we can apply the same reasoning to $\|\Omega^3(S^3 \vee S^3)\|_1$ to get a proof of syllepsis on general third loop spaces.

Our second application is to the definition of the sign homomorphism. We present a version of Cartier's argument (see also [5]). Synthetically, the sign homomorphism consists of a pointed map $BS_n \to_{\mathrm{pt}} BS_2$ from the type BS_n of n-element types to the type BS_2 of 2-element types. We describe what this function does to a general n-element type A. Note that the type $\binom{A}{2}$ of 2-element subsets of A is a finite type of cardinality $\binom{n}{2}$. Importantly, because the symmetric group S_2 on two elements is abelian, BS_2 is a spectrum, and so has unordered addition. Hence the sum of all 2-element subsets of A is a well-defined element of BS_2. We may think of this 2-element type as the type of *orientations* of A. The map sending A to its type $\mathsf{or}(A)$ of orientations is the (delooping of the) sign homomorphism.

For example, we have $\mathsf{or}(A) = A$ when A has cardinality 2, since then A is the unique 2-element subset of A, and the sum over a singleton is simply that value. We have $\mathsf{or}([n]) = [2]$, since each 2-element subset of $[n]$ has a canonical element given by the minimum, and an inhabited 2-element set is $[2]$, the basepoint or zero of BS_2, and any indexed sum of zeroes is zero. We similarly have $\mathsf{or}(A + B) = \mathsf{or}(A) + \mathsf{or}(B)$, where $A + B$ means coproduct of finite types and $\mathsf{or}(A) + \mathsf{or}(B)$ means addition in BS_2: any 2-element subset of $A + B$ is either a subset of A, a subset of B, or given by an element of A together with an element of B. In the third case, we can canonically pick an element (say, the one in A), so this third term does not contribute to the sum describing $\mathsf{or}(A + B)$.

More generally, we have $\mathsf{or}((a : A) \times B(a)) = \mathsf{or}(A') + \sum_{a:A} \mathsf{or}(B(a))$ where A' is the subset of A consisting of those a for which $B(a)$ has odd cardinality. To see this, we first note that there are two kinds of 2-element subsets of $(a : A) \times B(a)$: those where the first components are the same, and those where they differ. The contribution of the first kind to $\mathsf{or}((a : A) \times B(a))$ is precisely $\sum_{a:A} \mathsf{or}(B(a))$. As for the second kind, such a 2-element subset is given by a 2-element subset I of A together with $b : (i : I) \to B(i)$. The 'underlying' type is simply I. Since the type $BS_2 \to_{\mathrm{pt}} BS_2$ is 0-truncated, the sum $\sum_{j:J} I$ depends only on the cardinality of J. Since S_2 has characteristic two, the sum moreover depends only on the parity of the cardinality of J. Because of this, the contribution of the second kind is given by those two-element subsets $I : \binom{A}{2}$ for which $(i : I) \to B(i)$ has odd cardinality, which is precisely those I that are actually subsets of A'. This completes the proof of the stated identity. It seems to be closely related to the notion of graded commutativity.

Our third application is to the Barratt–Priddy–Quillen theorem. This concerns the sphere spectrum $\mathbb{S}$, which is freely generated by an element $e : \mathbb{S}_0$ of its underlying type. The Barratt–Priddy–Quillen theorem describes a relationship between $\mathbb{S}$ and the type of finite sets, mediated

[3]This is an unpublished result of Kapulkin and Sattler: given a closed term of type $\|A\|_{-1}$, there is also a closed term of A.

by a map $\mathsf{FinSet} \to \mathbb{S}_0$ from finite sets to the underlying type of $\mathbb{S}$. We can describe this map in terms of unordered addition: it sends $I : \mathsf{FinSet}$ to $\sum_{i:I} e$. Many properties of this map are readily verified, such as the fact that it respects addition. One expects that the action on π_1 corresponds to the sign homomorphism $S_n \to S_2$, under the equivalence $\pi_1(\mathbb{S}) \simeq S_2$ [2].

We can also describe the associated map from the delooping BS_∞ of the infinite symmetric group. First we give a description of BS_∞ as a higher inductive type: it is generated by terms $\iota(A) : BS_\infty$ for $A : \mathsf{FinSet}$ and paths $q(A) : \iota(A) = \iota(A+1)$. This definition, which is uniform in the cardinality of A, can also be expressed as a sequential colimit, which shows that BS_∞ is actually 1-truncated [3]. Next, we define a map $BS_\infty \to_{\mathrm{pt}} \mathbb{S}_0$ by $\iota(A) \mapsto (\sum_{a:A} e) - |A|e$, where $|A|$ denotes the cardinality of A. The action on paths of $q(A)$ is obtained by writing $\sum_{a:A+1} e = e + \sum_{a:A} e$ and $|A+1|e = e + |A|e$. The factorisation of this map through the path-component $\widehat{\mathbb{S}_0}$ of $\mathbb{S}_0$ at the basepoint should be an acyclic map $BS_\infty \to \widehat{\mathbb{S}_0}$. We do not know if the acyclicity of this map can be established in homotopy type theory.

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